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Moiré Patterns: Their Application to Refractive Index and Refractive Index Gradient Measurements

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The general theory of moiré patterns obtained from parallel rulings and concentric circles is presented. Near superposition of a regular ruling on to parallel rulings of variable spacings results in a curved moiré pattern which is functionally related to the deviation in spacings. When two figures consisting of uniformly spaced concentric circles are overlapped, the resulting moiré patterns are hyperbolas defined by the center-to-center distance of the figures and the inter-ring spacing.

Since very small relative displacements of the figures result in large changes in the moiré pattern, this technique can be a sensitive detector of minute changes in refractive index and in refractive index gradient which bring about apparent relative displacement of the figures. The technique is demonstrated for the case of a constant refractive index gradient and for a variable refractive index gradient as encountered in diffusion measurements. Birefringence and dispersion can also be determined utilizing the moiré method. Demonstration of the use of the moiré technique for the evaluation of lenses is also presented.

INTRODUCTION

MOIRÉ patterns most commonly refer to those patterns which one observes when two similar screens or sets of rulings are nearly superposed. As is considered in this paper, more general repetitive markings can also exhibit moiré patterns. Rayleigh[1] analyzed the case of two identical diffraction gratings placed in nearly parallel superposition. He appreciated the fact that this phenomenon could readily determine imperfections in gratings and the technique of testing gratings by this means has been investigated in detail at the National Physical Laboratory, Teddington.[2] Since very small relative displacements of the rulings is manifested in large movements of the moiré fringes, the technique can be utilized to measure extremely small movements.[2,3]

This powerful technique has not been utilized for the determination of relative displacements of the image of rulings caused by varying refractive index or by bending of light. The purpose of the present paper is to demonstrate that the moiré pattern technique provides an extraordinarily simple means of determining slight variations in refractive index and refractive index gradient. For these purposes it is necessary to consider the geometry for more general cases than merely that of identical rulings of parallel lines.

GEOMETRY OF MOIRÉ PATTERNS

The case of most interest for the present purpose is that of two gratings of parallel lines, one grating being of uniform spacings and the other having a uniform spacing differing from the first ruling or having spacings varying systematically. Also of interest is the case of two figures consisting of concentric circles.

Consider first the case of two rulings, one of which has equidistant spacings, a, (the a ruling), over which is placed at an angle θ a ruling of parallel lines whose spacing b, (the b ruling), is different from a. In Fig. 1 the two different rulings making an angle θ with each other form lines (moiré fringes) connecting points of

[1] Lord Rayleigh (J. W. Strutt), Phil. Mag. 47, 81, 193 (1874).
[2] J. Guild, *The Interference Systems of Crossed Diffraction Gratings* (Oxford University Press, Oxford, England, 1956).
[3] R. V. Jones and J. C. S. Richards, J. Sci. Instr. 36, 90 (1959); M. J. C. Flude, K. J. Habell, and A. Jackson, *ibid.* 38, 445 (1961).

intersection. The fringes are equidistant (spacing d) parallel lines represented by dotted lines in the figure and the angle which the pattern makes with the b rulings is given by φ.

An examination of the figure shows that

$$\sin\varphi = b \sin\theta / (a^2+b^2-2ab\cos\theta)^{\frac{1}{2}} \quad (1)$$

and

$$d = ab/(a^2+b^2-2ab\cos\theta)^{\frac{1}{2}}. \quad (2)$$

This latter expression can also be obtained by vectorial arguments.[4] For identical rulings, i.e., $a=b$, Eq. (2) reduces to

$$d = a/2\sin(\theta/2) \quad (3)$$

—that is, Bragg's law for first-order diffraction by a wave of wavelength a falling on a lattice of spacing d where θ is twice that chosen by Bragg. Here the moiré pattern established in the Bragg condition is equivalent to the crystallographic plane.

Now consider the case in which the displacement of the b rulings varies in a continuous manner along the Y direction. For example, in Fig. 2(a) the displacements of the b rulings from the positions for the original uniform ruling vary as a Gaussian function of Y. Since b differs progressively from a, if a regular ruling [Fig. 2(b)] is placed over this b ruling the moiré pattern results in the curve given in Fig. 2(c). In order for the moiré pattern to be a true representation of the displacement function the moiré curve should be referred to the Y axis. In other words, the uniform spacing of the a ruling should be equal to $b'\cos\theta$ where b' is the spacing of the b ruling in the uniform portion as shown in Fig. 2(b). For practical purposes it is sufficient to make the a rulings slightly smaller than the uniform b rulings and adjust θ so that the moiré pattern in the uniform portions lies parallel to the Y axis.

Two identical rulings of uniformly spaced circles whose centers are displaced exhibit moiré patterns of radiating hyperbolas as shown in Fig. 3. This arises from the fact that the moiré curve is the locus of a point that moves so that the difference of its distances from the centers of the circles is constant and describes a hyperbola. The foci of the hyperbolas are given by $n\lambda/2$ and $[(s/2)^2-(n\lambda/2)^2]^{\frac{1}{2}}$ where n is an integer, λ is the intercircle spacing, and s is the distance between the centers of the circles lying on the Y axis. These hyperbolas in-

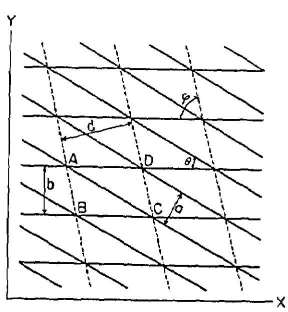

FIG. 1. Diagram of moiré fringes (dotted lines of spacing d) produced from two linear parallel rulings (spacing a and b). Note $AB^2=AC^2+BC^2-2AC\times BC\cos\theta$, and since area of $ABCD=AB\times d=AC\times a=BC\times b$ then $d^{-2}=a^{-2}+b^{-2}-2a^{-1}b^{-1}\cos\theta$.

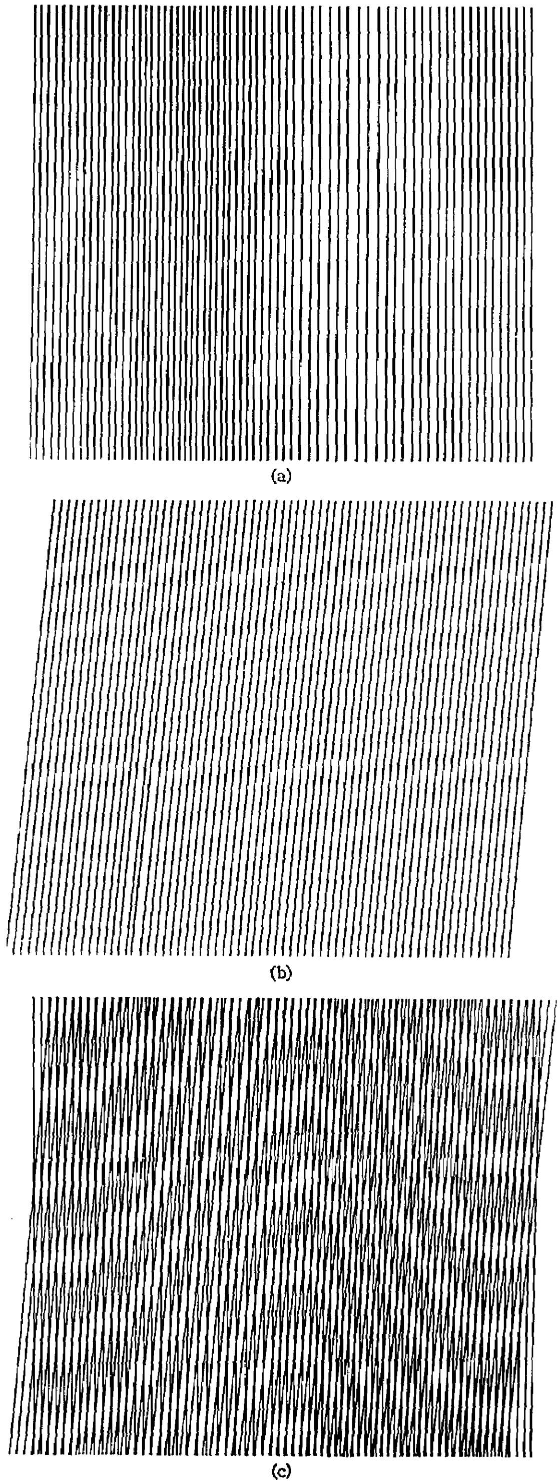

FIG. 2. Gaussian moiré pattern. (a) Displacements of b ruling following error function of vertical axis. (b) Corrected a ruling. (c) Superposition of (a) and (b).

[4] G. L. Rogers, Proc. Phys. Soc. (London) 73, 142 (1959).

tersect a line lying parallel to the Y axis at a position of y given by

$$y^2=[(n\lambda)^2/s^2-(n\lambda)^2]x^2+(n\lambda/2)^2, \qquad (4)$$

where x is the distance between the line of intersection and the Y axis. Michelson[5] has drawn a similar figure to illustrate the use of the Huyghens–Fresnel principle to explain Young's experiment for interference from two pinholes illuminated by a point light source. The final formula, which one obtains, namely, that $\sin\alpha=n\lambda/s$, where α is the angle of interference (note: $y/x=\tan\alpha$), neglects the second term on the right in Eq. (4). In other words, the standard formula for interference is actually applicable only to the asymptotes of the hyperbolas and presupposes that x is very much greater than s for a given value of $n\lambda$.

It is of interest to note that a radiating moiré pattern is observed by eye when the center of a single concentric circle pattern is displaced rather quickly. Evidently the moving moiré pattern is produced by the overlapping of the afterimage of the figure with the immediate image of the figure. This moiré pattern rotates with the speed of rotation at which the center is moved around and is more easily discernible than the subtle effects observed with stationary figures of this kind.[6]

MOIRÉ PATTERNS IN REFRACTOMETRY

Changes in refractive index or changes in refractive index gradient are manifested in a displacement of the image of a ruling and hence, using another ruling, this displacement can be observed as a change in the moiré pattern. A simple refractometer could consist of a rectangular cell whose opposite walls are two identical line rulings and the moiré pattern for a reference material is observed at some oblique angle. On introduction of another material between the plates the moiré pattern will shift in position relative to that which it had formerly. Hence, refractive index measurements can be made as sensitive as those for delicate physical displacements. If the two cell walls have figures of uniformly spaced concentric circles, the moiré pattern would be of

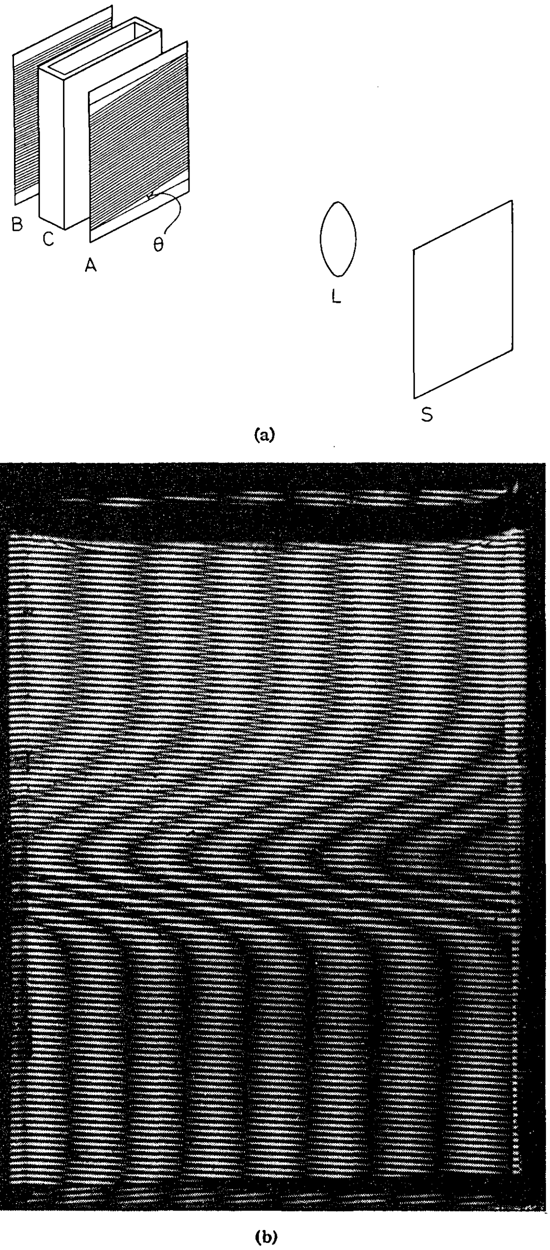

FIG. 4. (a) Diagrammatic sketch of diffusion apparatus. (b) Moiré pattern formed on the screen (S) in Fig. 4(a). Cell (C) thickness 1 cm; spacing of ruling (B) 0.2 mm; angle θ of ruling (A) 5°; focal length of lens (L) 150 mm. Diffusion curve for aqueous solutions of 60% glycerol against 50% glycerol.

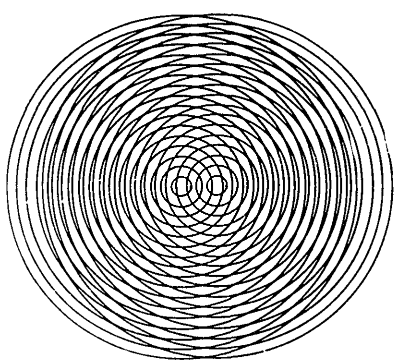

FIG. 3. Superposition of concentric circles.

the radial type and the angle α of the opening would be a measure of the refractive index of the material. Obviously, for materials having dispersion in the refractive index, colored moiré patterns will result on illumination with white light.

Dispersion measurements using the moiré technique can also be carried out on anisotropic materials. For example, if a rhomb of calcite is placed between two uniformly spaced linear rulings and the system is illuminated with white light, strongly colored moiré

[5] A. A. Michelson, *Studies in Optics* (University of Chicago Press, Chicago, Illinois, 1927), p. 11.
[6] D. M. MacKay, Nature **180**, 849, 1145 (1957).

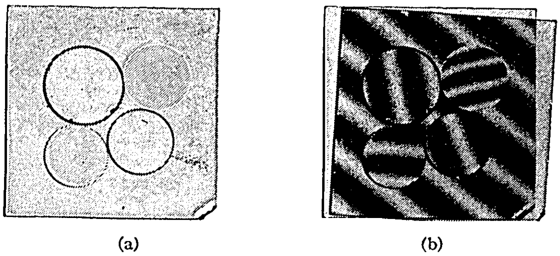

FIG. 5. Moiré patterns for lenses. (a) Figure produced by lenses placed on a regular ruling (upper left: plano convex $f=81$ mm, upper right: meniscus $f=-76$ mm, lower left: meniscus $f=-36$ mm, lower right: plano convex $f=114$ mm). (b) Moiré patterns produced by overlapping regular rulings on (a). Patterns for combinations of two lenses are achieved by placing the second lens on the appropriate portion of (a) and viewing through the regular rulings.

fringes are observed. There is one orientation of the crystal, namely, when the images of the ordinary and the extraordinary rays lie along the same ruling line, where the resulting moiré fringes exhibit no color. In other orientations of the crystal the colored moiré fringes indicate the dispersion of the extraordinary ray. For the determination of birefringence it is convenient to view a single figure containing equispaced concentric circles through the material and observe the hyperbolic moiré patterns caused by the double image of the figure (see Fig. 3).

When a regular parallel line ruling (b ruling) is viewed through a system containing a refractive index gradient the image of the ruling is distorted. In the case of constant index of refraction gradient, such as is achieved in a density gradient column,[7] the displacement of the b ruling, which is proportional to the value of the gradient, should be constant throughout the column. A simple procedure for determining the constancy of a gradient in a column is to place the b ruling perpendicular to the long axis of the column and view the system through the a ruling of slightly smaller spacings and rotate the a ruling until straight moiré lines run along the column. If straight lines are not achieved then there must be a variation in refractive index gradient. Any variation in the refractive index gradient is manifested by curved moiré lines which is a direct representation of the refractive index gradient curve, such as encountered in diffusion experiments.

A diagrammatic sketch of our diffusion apparatus is shown in Fig. 4(a). A diffusion cell (C) is placed in between two rulings (A and B), and the moiré pattern is formed on a screen (S) on which the images of A and B are focused by a lens (L). The diffusion cell is a rectangular cell with parallel walls separated by one centimeter. One solution occupying half the cell is carefully overlayed with another solution and the diffusion takes place in the vertical direction (Y axis). The ruling (B) with equispaced parallel lines with 0.2-mm spacings is placed on the cell wall facing the general illumination provided by a ground glass diffuser with a 60-W tungsten lamp behind it. The lines of this ruling (b ruling) are in the horizontal direction (X axis). When an identical ruling is placed on the other side of the cell one observed curved moiré patterns. This arises from the bending of light caused by the refractive index gradient within the cell[8] which distorts the image of the first ruling (B). In order that the moiré pattern be a true representation of the refractive index gradient-distance curve, the spacing and the orientation of the second ruling (A) must be corrected so that the image of the second ruling must fulfill the condition that the spacing is equal to $b' \cos\theta$ where b' is the spacing of the image of the first ruling in the undistorted portion [see Fig. 2(b)]. After making such an adjustment one obtains the result shown in Fig. 4(b) on the screen (S). In this moiré pattern, the base lines lie along the direction of diffusion and the diffusion coefficient may be calculated directly from the curve in the conventional manner.[9]

The height of the refractive index gradient-distance curve, namely, the magnification of the displacements in b ruling, can be adjusted by varying the angle θ; the smaller the angle the higher the magnification [see Eq. (2)]. When the refractive index gradient-distance curve consists of several peaks corresponding to more than one component, as encountered frequently in sedimentation or electrophoresis measurements, one can adjust the magnification high enough to evaluate the smallest peak. The overscaled higher peaks can be reconstructed by adding up the series of curves, since, as shown in Fig. 4(b), the moiré pattern consists of parallel identical refractive index gradient-distance curves. This technique of piling up curves also provides a sensitive means of evaluating polydispersity and concentration dependence of diffusion coefficient when applied for diffusion measurements. The accuracy of the measurements of the moiré method is essentially identical with that for Lamm's scale method.[9] However, the moiré method makes an instantaneous observation of diffusion process possible and avoids the tedious scale reading necessary for Lamm's scale method. The schlieren method[9] gives a direct view of the diffusion curve, but, since only one curve is obtained, the piling up process as described above to increase the sensitivity over the whole range of the observation is not possible to apply.

If a thin lens is interposed between two identical rulings where a moiré pattern has been established, the moiré pattern within the lens will be altered. The spacings of the image of the ruling behind the lens are changed [Fig. 5(a)]. Hence the moiré pattern shows a shifted angle and spacing according to Eqs. (1) and (2), respectively [Fig. 5(b)]. The change of the spacings of the image are determined by the focal length of the lens and hence the angular difference between the moiré lines produced in the lens image and the lines in the

[7] G. Oster and M. Yamamoto, Chem. Rev. 63, 257 (1963).

[8] O. Wiener, Ann. Phys. Chem. 49, 139 (1893).

[9] See, for example, W. Jost, *Diffusion in Solids, Liquids, and Gases* (Academic Press Inc., New York, 1960).

background is a measure of the focal length and the sign of the lens. A similar result is obtained by viewing with the lens a moiré pattern established between two nearly superposed but separated rulings. Any distortion of the moiré lines within the lens is indicative of geometrical aberrations. Chromatic aberration of a lens system is most easily demonstrated with white light by focusing with this lens system the image of a ruling on to another ruling and observing whether or not colored moiré patterns are obtained.

CONCLUSIONS

The moiré technique constitutes a direct-viewing method for the determination of refractive index gradient. In this respect it is much less expensive and far simpler than the existing methods, namely, the interference and schlieren techniques, which require strong light sources and precisely aligned multiple lens systems.

Because of the sensitivity of the moiré technique for slight changes in refractive index, the method could be applied to biological specimens immersed in water. The technique could be used to magnify such specimens without using lenses. One procedure would be to photograph the enlarged image of a fine ruling illuminated by a point source of light and from then on one merely interposes the specimen between the light source and the original ruling. A magnified image, in the form of a moiré pattern, of the specimen can be seen on the enlarged ruling.